Optimal Rating Design

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Introduction

- Information design is central to markets with asymmetric information
  - Peer-to-peer platforms: eBay and Airbnb
  - Regulating insurance markets: Community ratings in health insurance exchanges under ACA
  - Credit Ratings in consumer and corporate debt markets
  - Certification of doctors and restaurants

- Common feature:
  - Adverse selection and moral hazard
  - Intermediary observes information
  - Decides what to transmit to the other side

Introduction

- Key questions:
  - How should the intermediary transmit the information?
  - When is it optimal to hide some information?
  - How do market conditions affect optimal information disclosure?

The Model

- Competitive model of adverse selection and moral hazard
- Unit continuum of buyers
    - Payoffs:

$$q - t$$

      q: quality of the good purchased
      t: transfer
    - Outside option: 0

The Model

- Unit continuum of sellers
  - Produce one vertically differentiated product
  - Choose quality q
  - Differ in cost of quality provision

$$\text{Cost}: C(q,\theta); \theta \sim F(\theta)$$

  - Payoffs

$$t - C(q,\theta)$$

  - outside option: 0

The Model

Assumption. Cost function satisfies: $C_q > 0, C_\theta < 0, C_{qq} > 0, C_{\theta q} \leq 0$.

- First Best Efficient: maximize total surplus $q - C(q, \theta)$

$$C_q\left(q^{FB}(\theta), \theta\right) = 1$$

- Submodularity: $q^{FB}(\theta)$ is increasing in θ.
  - Higher θ's have lower marginal cost

Information Design

- Sellers know their θ and q
- An intermediary observes q and sends information about each seller to all buyers
  - Alternative: commit to a machine that uses q as input and produces random signal
- Intermediary chooses a *rating system:* (S, π)
  - S: set of signals
  - $\pi(\cdot|q) \in \Delta(S)$
- Buyers only see the signal by the intermediary
- Key statistic from the buyers perspective

$$\mathbb{E}[q|s]$$

Equilibrium

- Buyers costlessly search for products
  - There is a price for each signal: $p(s)$
- Buyers indifferent:

$$\exists u \geq 0, \quad \mathbb{E}[q|s] - p(s) = u \tag{1}$$

- Sellers payoff

$$q(\theta) \in \arg\max_{q'} \int p(s)\,\pi(ds|q') - C(q', \theta) \tag{2}$$

- Sellers participation: $\theta \in \Theta$

$$\int p(s)\,\pi(ds|q(\theta)) - C(q(\theta), \theta) \geq 0 \tag{3}$$

Equilibrium: $\left(\{q(\theta)\}_{\theta\in\Theta}, u, p(s)\right)$ that satisfy (1), (2) and (3).

Rating Design Problem

- The goal: find optimal (S, π) according to some objective
  - Pareto optimality of outcomes
  - Maximize intermediary revenue
  - etc.
- First step
  - What allocations are implementable for an arbitrary rating system
- Key object from seller's perspective: Expected price

$$\overline{q}(\theta) = \int p(s)\pi(ds|q(\theta)) = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q(\theta)\right] - u$$

We call it Signaled Quality.

Characterizing Rating Systems

- Start with discrete types $\Theta = \{\theta_1 < \cdots < \theta_N\}$ and distribution $F : \mathbf{f} = (f_1, \cdots, f_N)$
  - Boldface letters: vectors in $\mathbb{R}^N$
- Standard revelation-principle-type arugment leads to the following lemma

Lemma 1. If a vector of qualities, $\mathbf{q}$, and signaled qualities, $\overline{\mathbf{q}}$ arise from an equilibrium, then they must satisfy:

$$\overline{q}_N \geq \cdots \geq \overline{q}_1, q_N \geq \cdots \geq q_1$$
$$\overline{q}_i - C(q_i, \theta_i) \geq \overline{q}_j - C(q_j, \theta_i), \forall i, j$$

- Can ignore other deviations (off-path qualities): with appropriate out-of-equilibrium beliefs

Feasible Signaled Qualities

- Feasible signaled qualities: majorization ranking a la Hardy, Littlewood and Polya (1934)

Definition. $\mathbf{q}$ F- majorizes $\overline{\mathbf{q}}$ or $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$ if

$$\sum_{i=1}^{k} f_i \overline{q}_i \geq \sum_{i=1}^{k} f_i q_i, \forall k = 1, \cdots N-1$$
$$\sum_{i=1}^{N} f_i \overline{q}_i = \sum_{i=1}^{N} f_i q_i$$

- Note: majorization:
  - is equivalent to second order stochastic dominance
  - more suitable for our setup

Majorization: Main Result

Theorem. Consider vectors of signaled and true qualities, $\overline{\mathbf{q}}, \mathbf{q}$ and suppose that they satisfy

$$\overline{q}_1 \leq \cdots \leq \overline{q}_N, q_1 \leq \cdots \leq q_N$$

where equality in one implies the other. Then $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$ if and only if there exists a rating system (π, S) so that

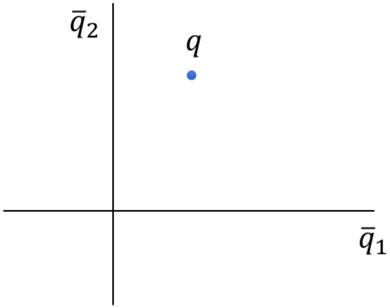
$$\overline{q}_i = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q_i\right]$$

Majorization: Proof of The Main Result

- First direction: If $\overline{q}_i = \mathbb{E}\left[\mathbb{E}\left[q|s\right]|q_i\right]$, then an argument similar to the above can be used to show that $\mathbf{q} \succcurlyeq_F \overline{\mathbf{q}}$.
    - If all states below k have separate signals from those above, then $\sum_{i=1}^{k} f_i \overline{q}_i = \sum_{i=1}^{k} f_i q_i$.
    - With overlap, $\sum_{i=1}^{k} f_i \overline{q}_i$ can only go up.

Majorization: Proof of The Main Result

- Second direction:
  - First step: show that the set of signaled qualities $\mathcal{S}$ is convex ▸ Proof
  - Second step: Illustration for $N = 2$.



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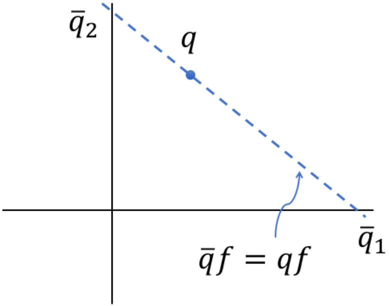

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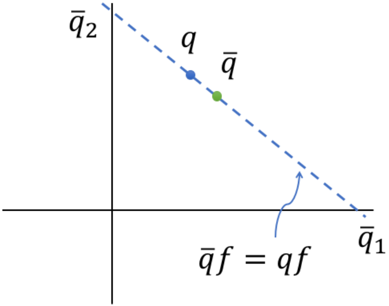

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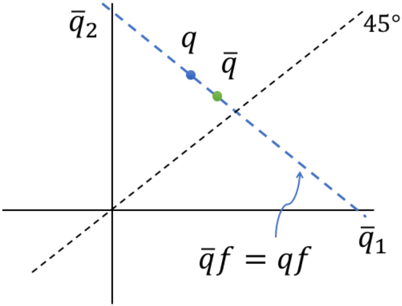

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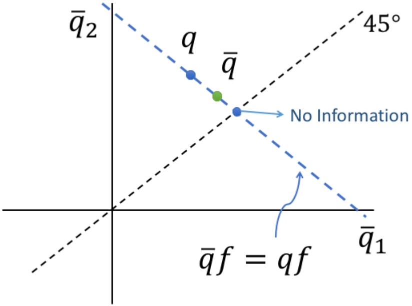

Majorization: Proof of The Main Result

- Second direction:
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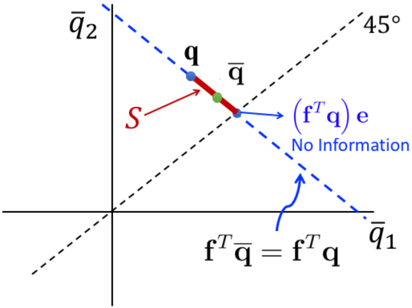


Majorization: Proof of The Main Result

- Second steps for higher dimensions:
  - For every direction $\boldsymbol{\lambda} \neq \mathbf{0}$, find two points in S, $\tilde{\mathbf{q}}$ such that

$$\boldsymbol{\lambda} \cdot \overline{\mathbf{q}} \leq \boldsymbol{\lambda} \cdot \tilde{\mathbf{q}}$$

    - If $\lambda_1/f_1 \leq \lambda_2/f_2 \leq \cdots \leq \lambda_N/f_N$, set $\tilde{\mathbf{q}} = \mathbf{q}$,
    - Otherwise, pool to consecutive states; reduce the number of states and use induction.
  - Since $\mathcal{S}$ is convex, separating hyperplane theorem implies that $\overline{\mathbf{q}}$ must belong to $\mathcal{S}$.

Majorization: Continuous Case

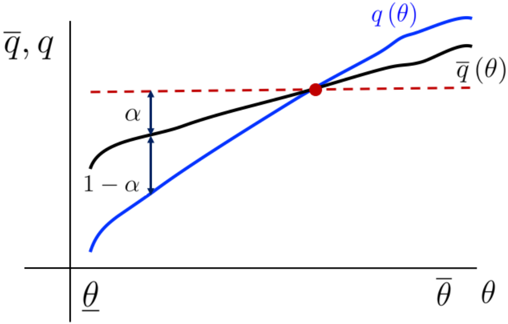
- We can extend the results to the case with continuous distribution
  - Discrete distributions are dense in the space of distributions.
  - Use Doob's martingale convergence theorem to prove approximation works
- We say $q(\cdot) \succcurlyeq_F \overline{q}(\cdot)$ if

$$\int_{\underline{\theta}}^{\theta} \overline{q}\left(\theta'\right) dF\left(\theta'\right) \geq \int_{\underline{\theta}}^{\theta} q\left(\theta'\right) dF\left(\theta'\right), \forall \theta \in \underline{\theta} = \left[\underline{\theta}, \overline{\theta}\right]$$

$$\int_{\underline{\theta}}^{\overline{\theta}} \overline{q}(\theta)\, dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} q(\theta)\, dF(\theta)$$

Constructing Signals

- Given $\overline{q}(\theta)$ and $q(\theta)$ that satisfy majorization: What is (π, S)?
- In general a hard problem to provide characterization of (π, S); Algorithm in the paper
- Example: *Partially revealing signal*



Optimal Rating Systems

- Pareto optimal allocations
- Approach:

$$\max \lambda_B u + \int \lambda(\theta)\,\Pi(\theta)\,dF(\theta)$$

subject to

(PC),(IC),(Maj)

- Our focus is on
  - $\lambda_B > \lambda(\theta)$: Buyer optimal
  - $\lambda_B \leq \lambda(\theta)$: increasing; higher weight on higher quality sellers
  - $\lambda(\theta)$: hump-shaped; higher weight on mid-quality sellers

Total Surplus

- Benchmark: First Best allocation
  - maximizes total surplus ignoring all the constraints

$$C_q\left(q^{FB}(\theta),\theta\right)=1$$

- Incentive constraint:

$$\overline{q}'(\theta)=C_q\left(q(\theta),\theta\right)q'(\theta)=q'(\theta)$$

- Set $\overline{q}(\theta)=q(\theta)$
  - Satisfies IC
  - Satisfies majorization
- Maximizing total surplus: full information about quality

Buyer Optimal Allocations

- Suppose that $\lambda_B > \lambda(\theta)$
  - Textbook mechanism design problem: all types have the same outside option; PC binding for $\underline{\theta}$
- Tradeoff: information rents vs. reallocation of profits
  - Want to allocate resources to the buyers
  - All higher quality types want to lie downward
- Reduce qualities relative to First Best

Buyer Optimal Allocations

Relaxed problem - w/o majorization constraint

$$\max u$$

subject to

$$\Pi'(\theta) = -C_\theta(q(\theta), \theta)$$

$$q(\theta) : \text{increasing}$$

$$u + \int_{\underline{\theta}}^{\overline{\theta}} \Pi(\theta)\, dF(\theta) = \int_{\underline{\theta}}^{\overline{\theta}} [q(\theta) - C(q(\theta), \theta)]\, dF(\theta)$$

$$\Pi(\theta) \geq 0$$

Proposition. A quality allocation $q(\theta)$ is buyer optimal if and only if it is a solution to the relaxed problem. Moreover, if the cost function $C(\cdot, \cdot)$ is strictly submodular, then a buyer optimal rating system never features a separation.

Buyer Optimal Allocations: Intuition

- The solution of the relaxed problem (with or without ironing)

$$C_q\left(q\left(\theta\right),\theta\right)<1$$

- Incentive constraint

$$\overline{q}'\left(\theta\right)=C_q\left(q\left(\theta\right),\theta\right)q'\left(\theta\right)$$

- $\overline{q}\left(\theta\right)$ flatter than $q\left(\theta\right)$: majorization constraint holds and is slack
    - If $C_q<1$ for a positive measure of types, no separation of qualities

Constructing Signals: Buyer Optimal

- When $\overline{q}(\theta)$ is flatter than $q(\theta)$ and majorization constraint never binds:
    - Finding signals is very straightforward: partially revealing signal

- Signal:

$$S = \{q(\theta) : \theta \in \Theta\} \cup \{\emptyset\}$$

$$\pi(\{s\}|q) = \begin{cases} \alpha(q) & s = q \\ 1 - \alpha(q) & s = \emptyset \end{cases}$$

    - Reveal quality or say nothing!

Buyer Optimal Rating

- Intuition:
  - Higher weight on buyers: Extract more from higher quality sellers
  - Underprovision of quality to avoid lying by the higher types
  - Some form of pooling is required to achieve this

Decreasing Welfare Weights

Corollary. If $\lambda(\theta)$ is decreasing in θ, then the majorization inequality is slack at the optimum. Furthermore, if $C(\cdot,\cdot)$ is strictly sub-modular, then the optimal rating system never features any separation.

High Quality Seller Optimal

- Suppose $\lambda(\theta)$ is increasing in θ
- Solution of the relaxed mechanism design problem satisfies

$$C_q(q(\theta), \theta) > 1$$

- IC:

$$\overline{q}'(\theta) = C_q(q(\theta), \theta) q'(\theta) > q'(\theta)$$

- Majorization inequality will be violated
  - Intuition: overprovision of quality to prevent low θ's from lying upwards; signaled quality must be steep

High Quality Seller Optimal

Proposition. Suppose that $\lambda(\theta)$ is increasing. Then optimal rating system is full information.

- Sketch of the proof:
  - Consider a relaxed optimization problem; replace IC with

$$\Pi(\theta) - \Pi(\underline{\theta}) \leq -\int_{\underline{\theta}}^{\theta} C_\theta\left(q\left(\theta'\right), \theta'\right) d\theta'$$

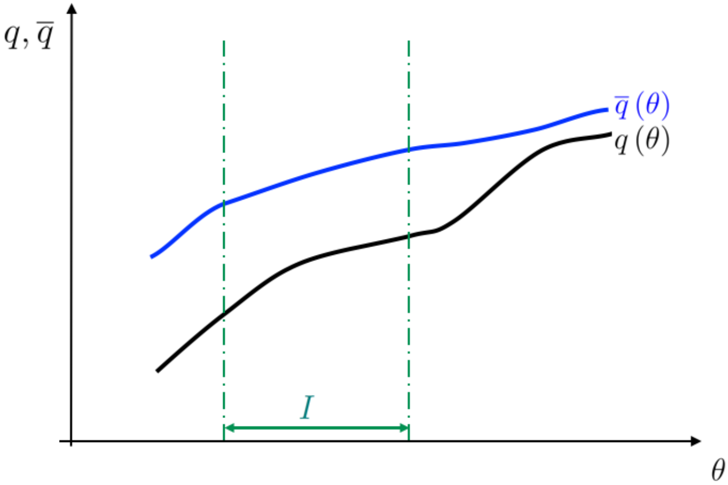
similar to restricting sellers to only lie upward

High Quality Seller Optimal

Proposition. Suppose that $\lambda(\theta)$ is increasing. Then optimal rating system is full information.

- Sketch of the proof: If majorization is slack for an interval I
  - relaxed IC must be binding: otherwise take from lower types and give it to higher types
  - overprovision of quality relative to FB, i.e., $C_q \geq 1$: if not:
    - increase q for those types; compensate them for the cost increase
    - distribute the remaining surplus across all types

High Quality Seller Optimal: Perturbation



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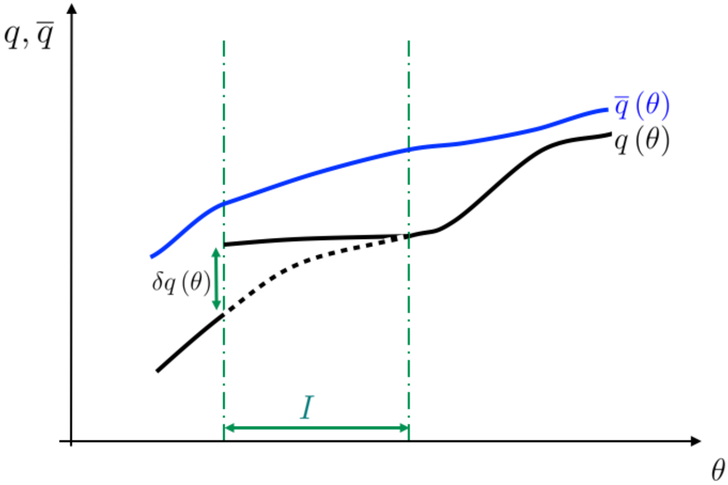

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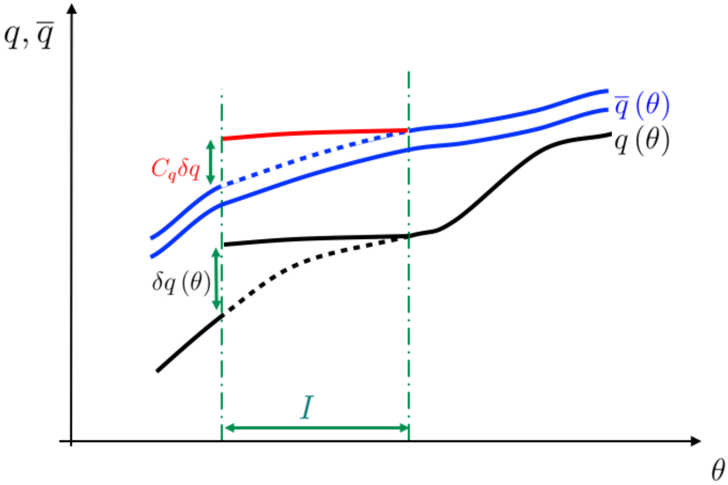


High Quality Seller Optimal

Proposition. Suppose that $\lambda(\theta)$ is increasing. Then optimal rating system is full information.

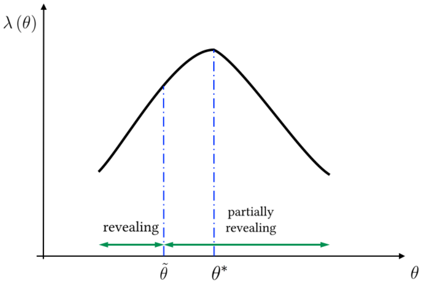
- Sketch of the proof:
  - Having majorization slack, incentive constraint binding and $C_q \geq 1$ is the contradiction

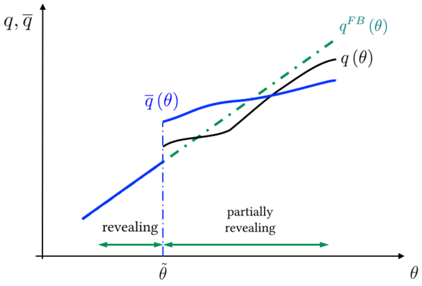
Mid Quality Seller Optimal

- Suppose $\lambda(\theta)$ is increasing below θ^* and decreasing above θ^*.

Proposition. Suppose that $\lambda(\theta)$ is hump-shaped. Then there exists $\tilde{\theta} < \theta^*$ such that for all values of $q \leq \lim_{\theta \nearrow \tilde{\theta}} q(\theta)$, the optimal rating system is fully revealing while it is partially revealing for values of q above $q\left(\tilde{\theta}\right)$. Finally, $q(\cdot)$ and $\overline{q}(\cdot)$ have a discontinuity at $\tilde{\theta}$.

Mid Quality Seller Optimal





Pareto Optimal Ratings

- General insight:
  - Cannot push profits towards higher qualities; at best should reveal all the information
  - Can use partially revealing to reallocate profits to lower qualities

Thank You!

Role of Entry

- Let's assume that the outside option of buyers is random: $v \sim G(v)$
- Outside option of sellers is π
- There will be an endogenous lower threshold θ for entry
- Everything is the same as before; all the results go through

Role of The Intermediary

- Suppose that the intermediary charges a flat fee
- Then problem is similar to the buyer optimal
- Partially revealing rating system is optimal

Related Literature

- Bayesian Persuasion: Kamenica and Gentzkow (2011), Rayo and Segal (2010), Gentzkow and Kamenica (2016), Dworczak and Martini (2019), Mathevet, Perego and Taneva (2019), ...
    - Characterize second order expectations + endogenous state
- Certification and disclosure: Lizzeri (1999), Ostrovsky and Schwartz (2010), Harbough and Rasmusen (2018), Hopenhayn and Saeedi (2019), Vellodi (2019), ...
    - Joint mechanism and information design
- (Dynamic) Moral Hazard and limited information/memory: Ekmekci (2011), Liu and Skrzpacz (2014), Horner and Lambert (2018), Bhaskar and Thomas (2018), ...
    - Hiding information is sometimes good for incentive provision

Convexity of $\mathcal{S}$

- Discrete signal space:

$$\overline{q}_i = \sum_s \pi\left(\{s\}\,|q_i\right) \frac{\sum_j \pi\left(\{s\}\,|q_j\right) f_j q_j}{\sum_j \pi\left(\{s\}\,|q_j\right) f_j}$$

- Alternative representation of the RS:

$$\tau \in \Delta\left(\Delta\left(\Theta\right)\right) : \mu_j^s = \frac{\pi\left(\{s\}\,|q_j\right) f_j}{\sum_j \pi\left(\{s\}\,|q_j\right) f_j}, \tau\left(\{\boldsymbol{\mu}^s\}\right) = \sum_j \pi\left(\{s\}\,|q_j\right) f_j$$

- Bayes plausibility

$$\mathbf{f} = \int_{\Delta(\Theta)} \boldsymbol{\mu} d\tau$$

- We can write signaled quality as

$$\overline{\mathbf{q}} = \text{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \mathbf{q} = \mathbf{A}\mathbf{q}$$

Convexity of $\mathcal{S}$

- The set $\mathcal{S}$ is given by

$$\mathcal{S} = \left\{ \overline{\mathbf{q}} : \exists \tau \in \Delta\left(\Delta\left(\Theta\right)\right), \int \boldsymbol{\mu} d\tau = \mathbf{f}, \overline{\mathbf{q}} = \operatorname{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau \right\}$$

- For any τ_1, τ_2 satisfying Bayes plausibility, i.e., $\int \boldsymbol{\mu} d\tau = \mathbf{f}$,their convex combination also satisfies BP since integration is a linear operator.
- Therefore

$$\begin{aligned} \lambda \overline{\mathbf{q}}_1 + \left(1-\lambda\right) \overline{\mathbf{q}}_2 = \lambda \operatorname{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau_1 + \\ \left(1-\lambda\right) \operatorname{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\tau_2 \\ = \operatorname{diag}\left(\mathbf{f}\right)^{-1} \int \boldsymbol{\mu}\boldsymbol{\mu}^T d\left(\lambda\tau_1 + \left(1-\lambda\right)\tau_2\right) \end{aligned}$$

- Since $\lambda\tau_1 + \left(1-\lambda\right)\tau_2$ satisfies BP, $\lambda \overline{\mathbf{q}}_1 + \left(1-\lambda\right) \overline{\mathbf{q}}_2 \in \mathcal{S}$

Back

Majorization: Basic Properties

- $\succcurlyeq_F$ is transitive.
- The set of $\overline{\mathbf{q}}$ that F-majorize $\mathbf{q}$ is convex.
- Can show that there exists a positive matrix $\mathbf{A}$ such that $\overline{\mathbf{q}} = \mathbf{A}\mathbf{q}$ where
$$\mathbf{f}^T\mathbf{A} = \mathbf{f}^T, \mathbf{A}\mathbf{e} = \mathbf{e}$$
with $\mathbf{e} = (1, \cdots, 1)$ and $\mathbf{f} = (f_1, \cdots, f_N)$.
- We refer to $\mathbf{A}$ as an F-stochastic matrix.
    - Set of F-stochastic matrices is closed under matrix multiplication.

Back

Constructing Signals

- One easy case: $\overline{q}(\theta)$ flatter than $q(\theta)$, i.e., $\overline{q}'(\theta) < q'(\theta)$
  - majorization constraint never binds.

- Signal:

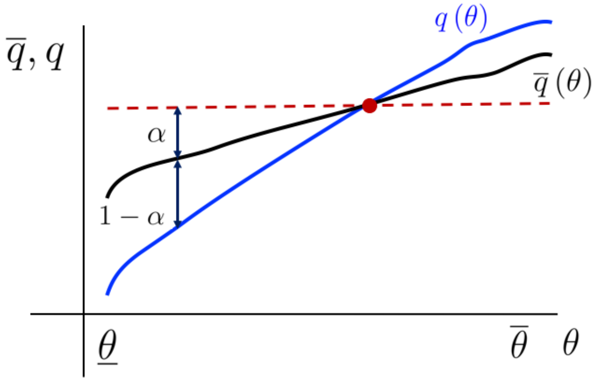
$$S = \{q(\theta) : \theta \in \Theta\} \cup \{\emptyset\}$$

$$\pi(\{s\} | q) = \begin{cases} \alpha(q) & s = q \\ 1 - \alpha(q) & s = \emptyset \end{cases}$$

  - Reveal quality or say nothing!

Non-separating signal

When $\overline{q}(\theta)$ is flatter than $q(\theta)$



Constructing Signals: Algorithm

- For the discrete case, we can give an algorithm to construct the signals (rough idea; much more details in the actual proof)
    1. Start from $\mathbf{q}$
    2. Consider a convex combination of two signals:
        2.1 Full revelation: $\pi^{FI}(\{q\}|q) = 1$
        2.2 Pooling signal: pool two qualities q_i and q_j

$$S = \{q_1, \cdots, q_N\} - \{q_i, q_j\} \cup \{q_{ij}\}$$

$$\pi^{i,j}(\{s\}|q) = \begin{cases} 1 & s = q, q \neq q_i, q_j \\ 1 & s = q_{ij}, q = q_i, q_j \end{cases}$$

        2.3 Send π^{FI} with probability α and $\pi^{i,j}$ with probability $1-\alpha$
    3. Choose α so that the resulting signaled quality has one element in common with $\overline{\mathbf{q}}$
    4. Repeat the same procedure on resulting signaled quality until reaching $\overline{\mathbf{q}}$ Back